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Rapport 4-82-5 Onderzoek HA-14

A general failure criterion for wood.

februari 1982 Ir. T.A.C.M. van der Put.

STEVIN-LABORATORIUM

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A GENERAL FAILURE CRITERION FOR WOOD

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Contents

0. Introduction

The concept of the yield surface is known from classical plasticity theory. For brittle fracture the meaning of the yield envelope is, that outside this envelope, continuous degradation of strength properties occur. One way to describe this surface is to look at particular mechanisms, f.i. plastic yielding, crack propagation, microbuckling, layer breaks and so on. This is shortly discussed in § 2. This approach is complicated by e.g. interacting mechanisms, and cracks in discrete interacting layers; so only a tendency can be given.
Another possibility is to describe the surface from test values. This is done for pine in § 1. The used tensor polynomial criterion meets the requirement of invariance, contains the properties of stress tensors, so can be regarded as a polynomial expansion of the yield surface. Therefore an accurate description is possible using as much terms as necessary. It is demonstrated for pine that one criterion can give all the variations in strength in the different planes by any stress combination, at any plane.
So far, the existing criteria apply only in certain circumstances and limited regions, and only in the main material planes.
The, at first glance curious behaviour of the off-grain-axis strength (p.e. fig. 8), is entirely determined by tensor-transformulations.
The general failure criterion contains some local strength increases and deviations from orthogonal strength behaviour. Therefore the critical distorsional energy theorem is only approximately true. For practice however, the oriëntation of the tangential and radial planes are not known, so a lower bound criterion has to be used that will be transverse isotropic depending on the weekest plane (see conclusion § 1.3. and § 2.6.).
The criterion for clear wood can also be used to investigate the influence of faults and knots in timber theoretically as next step.
It is intended to do tests in tri-axial compression and compression with low tension to measure the order of deviations from the simplest equation; the hardening properties and the influence of time.
This can be done by a multi-axial cell (see [15]).

1. Phenomenological failure criteria

1.1. General properties of initial yield surfaces

Because the many aspects of failure and the many possible mechanisms in different circumstances it seems to be useful to describe a yield surface as an inscribed envelope of those possible yield- and fracture-surfaces based on the appropriate, measured, independent strength components.

General expressions of the yield surface in strength tensors are mentioned in [1]: in the powerform:

$$(F_i - \sigma_i)^{\alpha} + (F_{ij}\,\sigma_i\,\sigma_j)^{\beta} + (F_{ijk}\,\sigma_i\,\sigma_j\,\sigma_k)^{\gamma} + \ldots = 1$$

that can be regarded as a general form on a polynomial basis known from invariant theory [9] or easier, and not less general in possibilities of fitting, in the tensor polynomial form:

$$F_i\,\sigma_i + F_{ij}\,\sigma_i\,\sigma_j + F_{ijk}\,\sigma_i\,\sigma_j\,\sigma_k + \ldots = 1 \quad (i, j, k = 1, 2, 3 \ldots 6),$$

that can be regarded as a general expansion of the real yield surface.

The polynomial basis can be derived from the following considerations of symmetry.

For wood the principal directions of strength may be regarded to be orthogonal and so the higher order terms F_{ijk} may be omitted and the simplest form of the failure surface in the stress space becomes:

$$F_i\,\sigma_i + F_{ij}\,\sigma_i\,\sigma_j = 1 \quad (i, j, k = 1, 2 \ldots 6) \tag{1}$$

For reasons of energetic reciprocity [5] $F_{ij} = F_{ji}$ $(i \neq j)$ and because wood can also be regarded to be orthotropic in the main planes, the interaction between the shear stresses can be disregarded $F_{ij} = 0$ $(i \neq j;$ $i, j = 4, 5, 6)$ so e.g. (1) is for a plane stress state in p.e. the 1-2 plane:

$$F_1\,\sigma_1 + F_2\,\sigma_2 + F_6\,\sigma_6 + F_{11}\,\sigma_1^2 + 2\,F_{12}\,\sigma_1\,\sigma_2 + F_{22}\,\sigma_2^2 + F_{66}\,\sigma_6^2 +$$

$$+ 2\,F_{16}\,\sigma_1\,\sigma_6 + 2\,F_{26}\,\sigma_2\,\sigma_6 = 1 \tag{1'}$$

For the same reasons of orthotropic symmetry in the main planes, thus expressed in the material axes (along the grain - tangential - radial directions) the shear strength has to be identical in positive and negative direction, so odd-order terms of σ_6 are zero and the coupling between normal- and shear strength vanish: $F_{16} = F_{26} = F_6 = 0$ and equation (1) becomes:

$$F_1 \sigma_1 + F_2 \sigma_2 + F_{11} \sigma_1^2 + 2 F_{12} \sigma_1 \sigma_2 + F_{22} \sigma_2^2 + F_{66} \sigma_6^2 = 1 \qquad (1'')$$

For a thermodynamic real surface (i.e. positive strain energy) the values: F_{ii} must be positive and also the failure surface cannot be open-ended (= hyperboloid) so the interaction terms are constrained to:
$F_{ii} F_{jj} > F_{ij}^2$ (no summation convention) or in (1"): $F_{11} F_{22} - F_{12}^2 > 0$ (2)
($F_{11} F_{22} = F_{12}^2$ is a parabolic surface and $F_{11} F_{22} < F_{12}^2$ is hyperbolic).
Eq. (1) can also be given in strain components:

$$G_i \varepsilon_i - G_{ij} \varepsilon_i \varepsilon_j = 1 \qquad (3)$$

with: $G_i = F_m S_{mi}$; $G_{ij} = F_{mn} S_{mi} S_{nj}$; S_{ij} = Elastic stiffness matrix being orthotropic too.
For the uniaxial tensile strength X along the grain (= parallel to the 1-axis) eq. (1") is: ($\sigma_6 = \sigma_2 = 0$).

$$F_{11} \sigma_1^2 + F_1 \sigma_1 = 1 \text{ with } \sigma_1 = X \text{ so } F_{11} X^2 + F_1 X = 1$$

and for uniaxial compression along the same direction: ($\sigma_1 = -X'$ as compression strength)

$$F_{11} (X')^2 - F_1 X' = 1$$

Solving the two equations in F_{11} and F_1 gives:

$$F_{11} = \frac{1}{XX'} \text{ and } F_1 = \frac{1}{X} - \frac{1}{X'}$$

In the same manner is:

$$F_{22} = \frac{1}{YY'} \text{ and } F_2 = \frac{1}{Y} - \frac{1}{Y'}$$

For pure shear ($\sigma_1 = \sigma_2 = 0$) eq. (1") is with $\sigma_6 = S$ as shear strength: $F_{66} = \frac{1}{S^2}$.

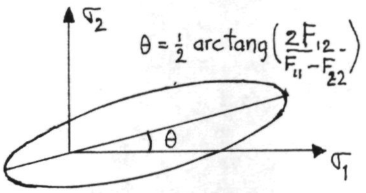
So for plane failure in the tangential plane (1") is:

$$\sigma_1\left(\frac{1}{X} - \frac{1}{X'}\right) + \sigma_2\left(\frac{1}{Y} - \frac{1}{Y'}\right) + \frac{\sigma_1^2}{XX'} + 2\ F_{12}\ \sigma_1\ \sigma_2 + \frac{\sigma_2^2}{YY'} + \frac{\sigma_6^2}{S^2} = 1 \qquad (4)$$

F_{12} has to be measured in bi-axial tests, because in an uni-axial test the influence of F_{12} is small and because F_{12} is very sensitive to small errors in X, Y and S. From the stability condition (2):

$$-\frac{1}{\sqrt{XX'\ YY'}} < F_{12} < +\frac{1}{\sqrt{XX'\ YY'}} \quad (\approx \frac{1}{XY} \text{ for wood})$$

In the criteria of Norris [2] or of Hill or Hoffman [3]: $F_{12} = \frac{-1}{2\ XY}$ or $F_{12} = -\frac{1}{2}\left(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}\right)$, and is not an independant quantity.
It is also suggested to ignore F_{12} i.e. $F_{12} = 0$ [4] for highly orthotropic materials (like wood is,giving errors $\lesssim \sim 20\%$, // grain).
Because the influence of F_{12} is small, analytical failure is almost independent of the type of failure criterion used for most types of test results.



F_{12} determines θ, the rotation of the failure-ellipsoid with respect to the material-axes. For wood this can be important for stresses $\perp$ grain.

Fig. 1. ($\sigma_6 = 0$).

Transformation of the strength tensors:
In the x', y' coördinates of fig. 2 the strength tensors are :

$$F_i = \begin{Bmatrix} F'_1 \\ F'_2 \\ F'_3 \\ F'_4 \\ F'_5 \\ F'_6 \end{Bmatrix} ; \quad F_{ij} = \begin{Bmatrix} F'_{11} & F'_{12} & F'_{13} & F'_{14} & F'_{15} & F'_{16} \\ & F'_{22} & F'_{23} & F'_{24} & F'_{15} & F'_{26} \\ & & F'_{33} & F'_{34} & F'_{35} & F'_{36} \\ & & & F''_{44} & F'_{45} & F'_{46} \\ & & & & F'_{55} & F'_{56} \\ \text{symmetry} & & & & & F'_{66} \end{Bmatrix}$$

The principal strength components are (in x, y):

$$F_i = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} ; \quad F_{ij} = \begin{Bmatrix} F_{11} & F_{12} & F_{13} & 0 & 0 & 0 \\ & F_{22} & F_{23} & 0 & 0 & 0 \\ & & F_{33} & 0 & 0 & 0 \\ & & & F_{44} & 0 & 0 \\ & \text{sym.} & & & F_{55} & 0 \\ & & & & & F_{66} \end{Bmatrix}$$

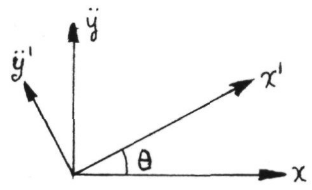


Fig. 2. Positive rotation about the main 3-axis (z-axis).

Transformation about the 3-axis gives:

$$F_1' = \frac{F_1 + F_2}{2} + \frac{F_1 - F_2}{2} \cos 2\theta$$

$$F_2' = \frac{F_1 + F_2}{2} - \frac{F_1 - F_2}{2} \sin 2\theta$$

$$F_6' = -(F_1 - F_2) \sin 2\theta$$

$$F_3' = F_3 \; ; \; F_4' = F_5' = 0$$

	invariant	cos 2θ	sin 2θ	cos 4θ	sin 4θ
F'_{11}	I_1	I_2	0	I_3	0
F'_{22}	I_1	$-I_2$	0	I_3	0
F'_{12}	I_4	0	0	$-I_3$	0
F'_{66}	$4I_5$	0	0	$-4I_3$	0
F'_{16}	0	0	$-I_2$	0	$-2I_3$
F'_{26}	0	0	$-I_2$	0	$+2I_3$
F'_{13}	I_6	I_7	0	0	0
F'_{23}	I_6	$-I_7$	0	0	0
F'_{36}	0	0	$-I_7$	0	0
F'_{44}	I_8	I_9	0	0	0
F'_{55}	I_8	$-I_9$	0	0	0
F'_{45}	0	0	I_9	0	0
F'_{33}	F_{33}	0	0	0	0

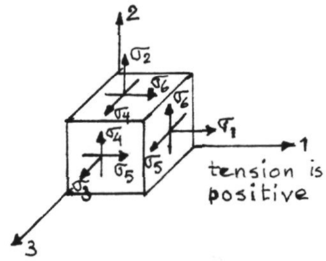


Fig. 3. Positive signs in right handed coördinate system.

$I_1 = (3F_{11} + 3F_{22} + 2F_{12} + F_{66})/8$
$I_2 = (F_{11} - F_{22})/2$
$I_3 = (F_{11} + F_{22} - 2F_{12} - F_{66})/8$
$I_4 = (F_{11} + F_{22} + 6F_{12} - F_{66})/8$
$I_5 = (F_{11} + F_{22} - 2F_{12} + F_{66})/8$
$I_6 = (F_{13} + F_{23})/2$
$I_7 = (F_{13} - F_{23})/2$
$I_8 = (F_{44} + F_{55})/2$
$I_9 = (F_{44} - F_{55})/2$

Read p.e.: $F'_{11} = I_1 + I_2 \cos 2\theta + I_3 \cos 4\theta$

Sign convention for shear:

If an outward normal of a plane points to a positive direction, the plane is positive, and if on a positive plane the stress component acts in the positive coördinate directions, this component is positive.

On a negative plane, the stress in negative coördinate direction is positive.
Outer the main directions there is a difference in positive and negative shear strength, so a sign convention is necessary.

Transformation is possible in two ways: The stress components can be transformed to the material-symmetry axes, so eq. (1") becomes:

$$F_1 \sigma_1' + F_2 \sigma_2' + F_{11}(\sigma_1')^2 + 2F_{12} \sigma_1' \sigma_2' + F_{22}(\sigma_2')^2 + F_{66}(\sigma_6')^2 = 1 \qquad (5)$$

Or the material symmetry axes can be rotated leaving the stress components unchanged so (1") is:

$$F_1' \sigma_1 + F_2' \sigma_2 + F_6' \sigma_6 + F_{11}' \sigma_1^2 + F_{22}' \sigma_2^2 + F_{66}' \sigma_6^2 + 2F_{12}' \sigma_1 \sigma_2 +$$
$$+ 2F_{16}' \sigma_1 \sigma_6 + 2F_{26}' \sigma_2 \sigma_6 = 1 \qquad (6)$$

1.2. Verification by test-results

To demonstrate the possibility of fitting test results to the simple failure criterion (1), strength values are taken from [6].

1.2.1. Shear perpendicular to the grain (rolling shear)

In most shear-test the strength is governed by the bending strength perpendicular to the grain and the high discontinuity peaks at the ends of the shearing plane. So the strength is a lower bound of a pre-cracked specimum. These effects are reduced in the tests mentioned in [6] page 904 by fitting the test-pieces precisely to the openings of the test-blocks. Only if the grain direction is parallel to the shear plane, it can be expected that additional stresses have a minor influence on the shear strength. So only these cases are considered here.

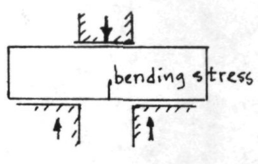


Fig. 4.

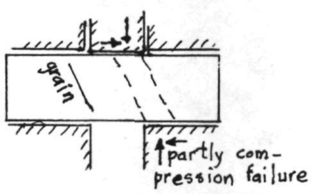


Fig. 5. excluded test.

For pure shear eq. (6) becomes ($\sigma_1 = \sigma_2 = 0$):

$$F_6' \sigma_6 + F_{66}' \sigma_6^2 = 1 \qquad (7)$$

In the main planes (tangential- and radial-plane), there is no difference in shear strength in one direction and the opposite direction so $F_6 = 0$ and if there is a shear stress in the tangential plane, the same stress is in the radial plane and the weakest plane controls the shearing strength so for $\theta = 0$ and $\theta = \frac{\pi}{2}$ eq. (7) is:

$$F_{66}\, \sigma_6^2 = 1 \rightarrow F_{66} = \frac{1}{27^2}$$

with $\sigma_6 \approx 27$ kgf/cm^2 as median of $\sigma_6 = 18\text{-}36$ kgf/cm^2 (see [6] page 906 for pine).

For $\theta = \frac{\pi}{4}$, a difference in positive and negative shear strength can be exptected, because for a shear stress in one direction there is a tensile stress in the tangential plane, and for shear in the opposite direction there is a tensile stress in the stronger radial plane.

In the double shear test at 45 degrees one plane has positive shear and the other, a negative shear and failure is first in the weakest plane. So only the negative shear-strength is measured. From [6] page 906 at $\theta \approx \frac{\pi}{4}$:

$$F'_6\, \sigma_6 + F'_{66}\, \sigma_6^2 = 1 \rightarrow -(F_1 - F_2)(\sin \tfrac{\pi}{2})\ 21{,}5 + \tfrac{4}{8}\,(F_{11} + F_{22} - 2F_{12} + F_{66})(21{,}5)^2$$

$$-\tfrac{4}{8}\,(F_{11} + F_{22} - 2F_{12} - F_{66})(\cos \pi)(21{,}5)^2 = 1$$

as $\sigma_6\ (\frac{\pi}{4}) = 21{,}5$ kgf/cm^2 as median of $\sigma_6 = 18\text{-}25$ kgf/cm^2

$$\therefore\quad -21{,}5\ \left(\frac{1}{X} - \frac{1}{X'} - \frac{1}{Y} + \frac{1}{Y'}\right) + (21{,}5)^2\ \left(\frac{1}{XX'} + \frac{1}{YY'} - 2F_{12}\right) = 1$$

On page 748 of [6] values are given of $X' = 60$ kgf/cm^2 als yield compression stress in the tangential plane ($\theta = 0$) and $Y' = 50$ kgf/cm^2 in the radial plane. The tensile strengths in the same planes can be taken to about 34 kgf/cm^2 and 45 kgf/cm^2 (page 670, 809)

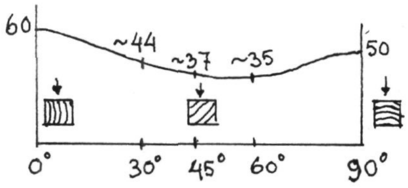


Fig. 6. pine compression ⊥

$$\rightarrow +\ 21{,}5\ \left(\frac{1}{34} - \frac{1}{60} - \frac{1}{45} + \frac{1}{50}\right) + (21{,}5)^2\left(\frac{1}{60.34} + \right.$$

$$\left. + \frac{1}{45.50} - 2F_{12}\right) = 1$$

$$+\ 0{,}226 + 0{,}432 - 924{,}5\ F_{12} = 1 \rightarrow F_{12} = -3{,}7\ .\ 10^{-4}$$

$$F_{12} > -\sqrt{F_{11} \cdot F_{22}} = -\sqrt{\frac{1}{50.34\ .\ 45.50}} = -4{,}67\ .\ 10^{-4}\ \text{(o.k.}$$

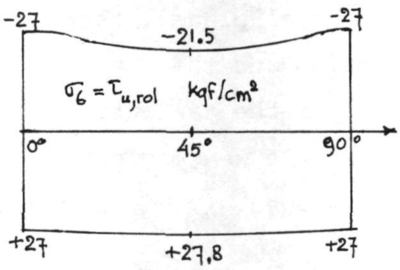


Fig. 7. Equal test- and theory-values.

With $F_{12} = -3,7 \cdot 10^{-4}$ the second root of the equation is $\sigma_6 = 27,8$ kgf/cm^2.
However, as stated before, the value of F_{12} is very sensitive for errors e.g. if $\sigma_6 = -20,5$ (i.s.o.: -21,5) → $F_{12} = +4,67 \cdot 10^{-4}$ i.e. just the upper bound of F_{12} and the positive root is +26,2 kgf/cm^2. F_{12} has to be measured in a bi-axial normal stress test, to give the best value. The existence of the higher positive shear than the negative shear is demonstrated in off-axes double shear tests where only one side is the testpiece is failing ([6] page 897).
For praxis it is necessary to give one value of rolling shear. The test results: σ_6 (θ = 0) = 18-36 kgf/cm^2; σ_6 (θ = 45) = 18-25 kgf/cm^2 show that this value can be based on some lower bound, p.e. $\sigma_6 = 18$ kgf/cm^2 or better, depending on the worst variance. In this case also bounds on the values of the tensile- and compression-strengths ⊥ are necessary:
$F_1 = F_2$ and $F_{12} = \dfrac{F_{11} + F_{22} - F_{66}}{2}$ or:

$$\frac{1}{X} - \frac{1}{X'} - \frac{1}{Y} + \frac{1}{Y'} = 0 \text{ and } F_{12} = \frac{1}{2}\left(\frac{1}{XX'} + \frac{1}{YY'} - \frac{1}{(\tau_{rol})^2}\right) \text{ with } F_{66} = \frac{1}{(\tau_{rol})^2}$$

eq. 7 becomes: $\tau \leqq \tau_{rol}$.

1.2.2. Uniaxial strengths ⊥ grain

In fig. 6 (from [6] page 748) values are given for the off-axis uniaxial compression strength perpendicular to the grain.
eq. (6) becomes with $\sigma_2 = \sigma_6 = 0$: $F'_{11}\,\sigma_1^2 + F'_1\,\sigma_1 = 1$ or:

$$\sigma_1\left(\frac{F_1 + F_2}{2} + \frac{F_1 - F_2}{2}\cos 2\theta\right) + \sigma_1^2\left(\frac{3F_{11} + 3F_{22} + 2F_{12} + F_{66}}{8} + \right.$$
$$\left. + \frac{F_{11} - F_{22}}{2} \cdot \cos 2\theta + \frac{F_{11} + F_{22} - 2F_{12} - F_{66}}{8} \cdot \cos 4\theta\right) = 1$$

For $\theta = 0^\circ$: $\sigma_o F_1 + \sigma_o^2 F_{11} = 1 \rightarrow \sigma_o(\frac{1}{X} - \frac{1}{X'}) + \sigma_o^2 \frac{1}{XX'} = 1 \rightarrow \sigma_{01} = X;\ \sigma_{02} = -X'$

For $\theta = 90^\circ$: $\sigma_{90} F_2 + \sigma_{90}^2 F_{22} = 1 \rightarrow \sigma_{90}(\frac{1}{Y} - \frac{1}{Y'}) + \sigma_{90}^2 \frac{1}{YY'} = 1 \rightarrow \sigma_{90} = Y;\ \sigma_{90} = -Y'$

For $\theta = 30^\circ$: $\sigma_{30}(\frac{3}{4} F_1 + \frac{F_2}{4}) + \sigma_{30}^2(\frac{9}{16} F_{11} + \frac{F_{22}}{16} + \frac{6}{16} F_{12} + \frac{3}{16} F_{66}) = 1$

For $\theta = 60^\circ$: $\sigma_{60}(\frac{1}{4} F_1 + \frac{3}{4} F_2) + \sigma_{60}^2(\frac{F_{11}}{16} + \frac{9}{16} F_{22} + \frac{6}{16} F_{12} + \frac{3}{16} F_{66}) = 1$

For $\theta = 45^{\circ}$: $\sigma_{45}\left(\frac{F_1}{2}+\frac{F_2}{2}\right)+\sigma_{45}^2\left(\frac{F_{11}}{4}+\frac{F_{22}}{4}+\frac{2F_{12}}{4}+\frac{F_{66}}{4}\right) = 1$

With: $F_1 = \frac{1}{X}-\frac{1}{X'}=\frac{1}{34}-\frac{1}{60}$; $F_2 = \frac{1}{Y}-\frac{1}{Y'}=\frac{1}{45}-\frac{1}{50}$; $F_{11}=\frac{1}{60 \times 34}$; $F_{22}=\frac{1}{50 \times 45}$;

$F_{66}=\frac{1}{27^2}$ and the upper bound of F_{12}: $F_{12} = 4{,}67 \cdot 10^{-4}$ (see § 1.2.1.).

The following values are found, as roots of the equations, given in fig. 8.

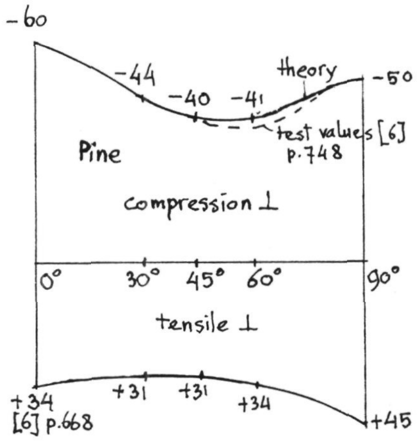


Fig. 8. Yield stresses.

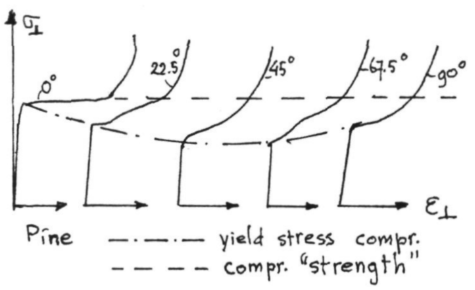


Fig. 9.

A better fit is possible by calculating the main tensile strengths X and Y from the equations inserting some measured values $-\sigma_{30}$, $-\sigma_{45}$ and $-\sigma_{60}$ with:

$$F_{12} = \alpha \sqrt{\frac{1}{XY\ X'Y}} \quad (\alpha \leqslant 1)$$

However, even with approximate values from incomparable tests, a good fit is possible. In [6] the tensile strength ⊥ is mostly taken to be 1/2 to 3/4 times the compression strength (being 60 kgf/cm^2 see below) so ~ 30 to 45 kgf/cm^2 and the radial tensile strength is stated to be about 1,5 times the tangential strength (≈ 45 kgf/cm^2). The value of 34 kgf/cm^2 is given in [6] as the best value of the tangential tensile strength. (Probably the physical conditions as moisture content, density, volume factor, are not very different in those tests).

In Fig. 9. ([6] page 721) it is seen that after first flow, hardening is occurring and after some equal plastic deformation the stresses are almost the same, independent of oriëntation.

So for porportional loading (in practice occurring) this constant stress can be taken as strength value. Then F_{12} must be bounded too, giving:

$$F_{12} = \frac{F_{11} + F_{22} - F_{66}}{2}$$

This condition is the same as in § 1.2.1. for one value of rolling shear strength and taking $F_1 = F_2$ because of § 1.2.1., eq. (6) becomes for this case:

$$\sigma_1\left(\frac{1}{X} - \frac{1}{X'}\right) + \sigma_1^2\left(\frac{1}{XX'}\right) = 1 \quad (X \approx \frac{1}{2} X' \approx 30 \text{ kgf/cm}^2)$$

So it is necessary to choose a constant lower bound of the tensile strength too.

In [6] page 809, probably the lower bounds were taken as given in fig. 8, so $X' = -40$ and $X \approx 30$ kgf/cm^2. In this case F_{12} is a small quantity and may be ignored because

$$F_{12} = \frac{F_{11} + F_{22} - F_{66}}{2} = F_{11} - \frac{F_{66}}{2} = \frac{1}{XX'} - \frac{1}{2\tau_{rol}^2} = \frac{1}{30x40} - \frac{1}{2\tau_{rol}^2} \approx 0 \quad \text{if}$$

$$\tau_{rol} \approx \sqrt{30.20} = 24,5 \text{ kgf/cm}^2$$

It is seen in fig. 7 that this value is close to 21,5 and 27 kgf/cm^2, at first flow.

1.2.3. Pure shear parallel to the grain

For a rotation about the 3-axis or the axis in the grain direction eq. (6) becomes for pure shear in that direction:

$$F'_{44}\ \sigma_4^2 = 1 \quad \text{with} \quad F'_{44} = \frac{F_{44} + F_{55}}{2} + \frac{F_{44} - F_{55}}{2} \cos 2\theta$$

Sor for

$$\theta = 0 \rightarrow F'_{44} = F_{44} \quad \text{and for } \theta = 90^\circ \rightarrow F'_{44} = F_{55}$$

In [6] page 906 and 907 the values are given as: $\sigma_5 = 100,5$ (89-112 kgf/cm^2) in the tangential plane, and $\sigma_4 = 114,5$ (110-119 kgf/cm^2) in the radial plane.

At $\theta = 45^\circ$: $\sigma_4\ (45^\circ) = 103$ kgf/cm^2 (93-113 kgf/cm^2).

Predicted from theory is $F'_{44}\ (45^\circ) = \frac{F_{44} + F_{55}}{2} = \frac{1}{2}\left(\frac{1}{(100,5)^2} + \frac{1}{(114,5)^2}\right) = \frac{1}{(106,8)^2}$

so $\sigma_4\ (45^\circ) = \frac{1}{\sqrt{F'_{44}\ (45^\circ)}} = 106,8$ kgf/cm^2.

This is close to the measured value of 103 kgf/cm^2.

More general $F'_{44} = \frac{F_{44} + F_{55}}{2} + \frac{F_{44} - F_{55}}{2} \cos 2\theta - \frac{F_{45}}{2} \sin 2\theta.$

So it is seen that the interaction value between σ_4 and σ_5: $F_{45} \approx 0$ as expected from general considerations (§ 1.1).

1.2.4. Shear strength parallel to the grain with compression perpendicular to the grain

The type of tests used in [6] give a higher shear strength than measured in [7]. This is explained in § 2.3.
Tests from [7] show a deviation from orthogonal strengths. In these tests, in the tangential plane, the influence of normal stress on the shear "strength" is small. In the radial plane there is an increasing shear "strength" with increasing compression stress normal to this plane (fig. 10). So coupling terms between σ_1 en σ_6 can not be neglected in this case. Because the shear strength in the main planes is independent of the sign of the stress, odd terms (p.e. F_6, F_{16}, F_{26}) disappear and higher order terms must be used, and the failure surface becomes:

$$F_i \sigma_i + F_{ij} \sigma_i \sigma_j + F_{ijk} \sigma_i \sigma_j \sigma_k = 1$$

For symmetry reasons (see § 1.1) $F_{ijk} = F_{ikj} = F_{jki} = F_{kij} = F_{kji}$. Further, the cubic terms F_{iii} are redundand and can be omitted.
So with even-order terms in σ_6 the equation becomes for plane stress:

$$F_1 \sigma_1 + F_2 \sigma_2 + F_{11} \sigma_1^2 + F_{22} \sigma_2^2 + F_{66} \sigma_6^2 + 2F_{12} \sigma_1 \sigma_2 + 3F_{112} \sigma_1^2 \sigma_2 +$$

$$+ 3F_{221} \sigma_2^2 \sigma_1 + 3F_{166} \sigma_1 \sigma_6^2 + 3F_{266} \sigma_2 \sigma_6^2 = 1$$

Because of minor interaction between σ_1 and σ_2 in the usual applied plane fracture tests, F_{112} and F_{221} can be neglected and in the tangential plane also F_{266} is small so there remain:

$$F_1 \sigma_1 + F_2 \sigma_2 + F_{11} \sigma_1^2 + F_{22} \sigma_2^2 + F_{66} \sigma_6^2 + 2F_{12} \sigma_1 \sigma_2 + 3F_{166} \sigma_1 \sigma_6^2 = 1 \quad (8)$$

This surface has to be closed; so for 2 collinear loading paths, there are only 2 distinct roots and taking the proportional loading path as:

$\sigma_1 = s_1\,\lambda;\ \sigma_2 = s_2\,\lambda;\ \sigma_6 = s_6\,\lambda$, the equation becomes:

$$3\lambda^3 s_1 s_6^2 F_{166} + \lambda^2 (F_{11} s_1^2 + F_{22} s_2^2 + 2F_{12} s_1 s_2 + F_{66} s_6^2) + \lambda (F_1 s_1 + F_2 s_2) - 1 = 0 \quad \text{or :}$$

$$a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_o = 0$$

Substitution of $\lambda = z - \frac{a_2}{3a_3}$ gives:

$$z^3 + 3pz + 2q = 0$$

with:

$$p = \frac{a_1}{3a_3} - \left(\frac{a_2}{3a_3}\right)^2 \quad ; \quad q = \left(\frac{a_2}{3a_3}\right)^3 - \frac{a_2\, a_1}{6a_3^2} + \frac{a_o}{2a_3}$$

For $p^3 = -q^2$ there are two equal roots and a third root

$$z_1 = -2\sqrt[3]{q} \text{ and } z_2 = z_3 = \sqrt[3]{q} \quad \text{so: } \lambda_1 = -2\sqrt[3]{q} - \frac{a_2}{3a_3} \text{ and } \lambda_2 = \sqrt[3]{q} - \frac{a_2}{3a_3}$$

For $p^3 < -q^2$ there are 3 different real roots (p, is negative than) with the substitution $kx = z$ or $k^3 x^3 + 3pkx + 2q = 0$ and $k = 2\sqrt{|p|}$ this becomes: $x^3 - \frac{3}{4} x + \frac{q}{4\sqrt{|p|^3}} = 0$ and has the form:

$$\sin^3 \alpha - \frac{3}{4} \sin \alpha + \frac{1}{4} \sin 3\alpha = 0$$

So $z = k \sin \alpha$ and $\sin 3\alpha = \frac{q}{\sqrt{|p|^3}} \rightarrow z = 2\sqrt{|p|} \cdot \sin \frac{1}{3} \arcsin \left(\frac{q}{\sqrt{|p|^3}}\right)$ and

$$\lambda = -\frac{a_2}{3a_3} + 2\sqrt{|p|} \cdot \sin \left(\frac{1}{3} \arcsin \left(\frac{q}{|p|\sqrt{|p|}}\right)\right)$$

From the 3 roots $(0 \leqslant \alpha \leqslant 2\pi)$ the negative one, and the smallest positive one can be on the fracture plane.

So $p^3 + q^2 \leqslant 0$ gives a bound of F_{166}. The equal sign may be approached from the lower side to retain a closed surface as can be seen in the following example.

For $\sigma_2 = 0$ eq. (8) becomes:

$$F_1 \sigma_1 + F_{11} \sigma_1^2 + F_{66} \sigma_6^2 + 3F_{166} \sigma_1 \sigma_6^2 = 1 \qquad (9)$$

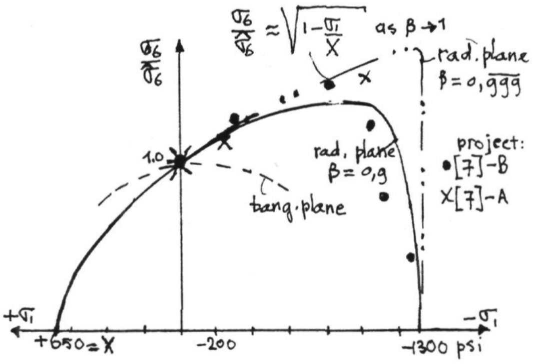


Fig. 10.

Fitting this curve to the values of [7] project A is only exactly possible when the tensile strength ⊥ and the compression strength ⊥ are known.
From proj. B: -1300 psi is assumed. For compressional- and $\sim\frac{1}{2} \times 1300 = 650$ psi for tensile-strength (as usual taken, § 1.2.2).
So eq. (9) becomes: ($\hat{\sigma}_6 = S = \frac{1}{\sqrt{F_{66}}}$)

$$\frac{\sigma_6}{\hat{\sigma}_6} = \sqrt{\left(\frac{\left(1 - \frac{2\sigma_1}{1300}\right)\left(1 + \frac{\sigma_1}{1300}\right)}{1 + \beta\,\frac{\sigma_1}{1300}}\right)} \qquad (10)$$

With:

$$\beta = \frac{3F_{166} \times 1300}{F_{66}} = 3900\ F_{166} \cdot \hat{\sigma}_6^2$$

It is seen from fig. 10 that with β somewhere between 0,9 and $0,\overline{99}$ (→ 1), a good fit is possible.
For β ≈ 0,9 the fit is even reasonable for the values of proj. B, demonstrating that the influence of σ_2 is probably small.
For proj. B eq. 8 must be used with: $\sigma_1 = \sigma_{max} \cos^2 \theta$; $\sigma_2 = \sigma_{max} \sin^2 \theta$; $\sigma_6 = \sigma_{max} \sin \theta \cos \theta$, but the strengths for compression an tension in the tangential and radial plane are not given in [7] and construction from the measured values that are given (see fig. 10: points) will probably introduce great errors in these strength values and in F_{166}.

Rotating eq. (9) about the 3-axis for 90° gives: ($\sigma_2 = 0$)

$$F_{66}\,\sigma_6^2 = 1 \text{ or } \sigma_6 = \hat{\sigma}_6$$

If $\sigma_2 \neq 0$ in the tangential plane but $\sigma_1 = 0$, the fracture surface is:

$$F_2\,\sigma_2 + F_{22}\,\sigma_2^2 + F_{66}\,\sigma_6^2 = 1$$

getting an elliptic form (fig. 10 ----) eq. (10) of the radial plane approaches the parabolic form (with cut off at -X'), known from fracture mechanics. (Also the low values of σ_6, much lower than $|-\sigma_1|$ in stead of much higher, [6] indicate initial cracks, see § 2.3).

1.2.5. Uniaxial off-grain-axis strength

So far all rotations were around the grain axes. Taking the 3-axis of rotation in the tangential- or radial-direction, the same general fracture equations apply as given in § 1.2.2.

Usual the tangential fracture plane is regarded, giving a lower bound of the strength (or being the weakest plane).
The existing criteria are given in the main plane, using the transformation of eq. (5).
For uniaxial stress is:

$$\sigma_1^1 = \sigma_m \cos^2 \theta; \; \sigma_2^1 = \sigma_m \sin^2 \theta; \; \sigma_6^1 = \sigma_m \sin \theta \cos \theta$$

and eq. (5) becomes:

$$\sigma_m \cos^2 \theta \left(\frac{1}{X} - \frac{1}{X'}\right) + \sigma_m \sin^2 \theta \left(\frac{1}{Y} - \frac{1}{Y'}\right) + \frac{\sigma_m^2 \cos^4 \theta}{XX'} + 2F_{12}\, \sigma_m^2 \sin^2 \theta \cos \theta +$$

$$+ \frac{\sigma_m^2 \sin^4 \theta}{YY'} + \frac{\sigma_m^2 \sin^2 \theta \cos^2 \theta}{s^2} = 1 \qquad (11)$$

The first two terms can be written:

$$\frac{\sigma_m \cos^2 \theta}{X} + \frac{\sigma_m \sin^2 \theta}{Y} - \left(\frac{\sigma_m \cos^2 \theta}{X'} + \frac{\sigma_m \sin^2}{Y'}\right)$$

and because:

$$\frac{\sigma_m \cos^2 \theta}{X} + \frac{\sigma_m \sin^2 \theta}{Y} = 1$$

(Hankinsins formula see [6] page 809; 669; 747) or:

$$\frac{\sigma_m \cos^2 \theta}{X'} + \frac{\sigma_m \sin^2 \theta}{Y'} \approx 1$$

for compression, F_{12} is known from (11).
Taking the square of both terms, the last equation for compression is:

$$\frac{\sigma_m^4 \cos^4 \theta}{(X')^2} + \frac{2\sigma_m^2 \cos^2 \theta \sin^2 \theta}{(Y')(X')} + \frac{\sigma_m^2 \sin^4 \theta}{(Y')^2} \approx 1$$

This must be approximately equivalent to the Norris equation for this case:

$$\frac{\sigma_m^2 \cos^4\theta}{(X')^2} - \frac{\sigma_m^2 \cos^2\theta \sin^2\theta}{X'\,Y'} + \frac{\sigma_m^2 \sin^2\theta \cos^2\theta}{S^2} + \frac{\sigma_m^2 \sin^4\theta}{(Y')^2} = 1$$

So:

$\frac{1}{S^2} \approx \frac{3}{X'\,Y'}$, and in the same way $\frac{1}{S^2} \approx \frac{3}{XY}$ for tension.

The value $S = \sqrt{\frac{X'\,Y'}{3}} = 0{,}58\sqrt{X'\,Y'} \approx 0{,}6\sqrt{X'\,Y'}$ is measured in [8].
Taking the product of both Hankinson's formula's,

$$\left(\frac{\sigma_m \cos^2\theta}{X} + \frac{\sigma_m \sin^2\theta}{Y} - 1\right)\left(\frac{\sigma_m \cos^2\theta}{X'} + \frac{\sigma_m \sin^2\theta}{Y'} + 1\right) = 0$$

being the condition for failure in tension or compression, so:

$$\frac{\sigma_m^2 \cos^4\theta}{XX'} + \frac{\sigma_m^2 \sin^2\theta \cos^2\theta}{X'\,Y} + \frac{\sigma_m^2 \sin^4\theta}{YY'} + \sigma_m \cos^2\theta \left(\frac{1}{X} - \frac{1}{X'}\right) +$$

$$+ \sigma_m \sin^2\theta \left(\frac{1}{Y} - \frac{1}{Y'}\right) + \frac{\sigma_m^2 \sin^2\theta \cos^2\theta}{XY'} = 1$$

then this has to be equivalent to eq. (11):

$$\frac{\sigma_m^2 \cos^4\theta}{XX'} + \left(2F_{12} + \frac{1}{S^2}\right)\sigma_m^2 \sin^2\theta \cos^2\theta + \frac{\sigma_m^2 \sin^4\theta}{YY'} + \sigma_m \cos^2\theta \left(\frac{1}{X} - \frac{1}{X'}\right) +$$

$$+ \sigma_m \sin^2\theta \left(\frac{1}{Y} - \frac{1}{Y'}\right) = 1$$

So:

$$2F_{12} + \frac{1}{S^2} \approx \frac{1}{X'\,Y} + \frac{1}{XY'}$$

For clear wood, mostly: $X \approx 2X'$ and $Y' \approx 2Y$ are taken for the strength so $XY = 2X'\,Y'/2 = X'\,Y'$ and $2F_{12} \approx \frac{1}{X'\,Y} + \frac{1}{XY'} - \frac{1}{S^2} \approx \frac{2}{XY} + \frac{1}{2XY} - \frac{3}{XY} = -\frac{1}{2XY}$

or, as a first approximation, F_{12} is in the order of:

$$F_{12} \approx -\frac{1}{4XY} \approx -\frac{1}{4X'\,Y'} \quad \left(> \frac{-1}{\sqrt{XX'\,YY'}} \approx \frac{-1}{X'\,Y'} \text{ for stability}\right)$$

Because of the strong, orthotropy this value is small in the main planes.

Therefore the older Norris equation is a better approximation of the strength. In that case:

$S \approx \sqrt{\frac{X' Y'}{2}}$ and $F_{12} \approx + \frac{1}{4X' Y'}$ (F_{12} gets the opposite sign, but remains small)

For off-axis strengths the equation for uni-axial stress is:

$$F'_{11} \sigma_1^2 + F'_1 \sigma_1 = 1 \qquad \text{(as in § 1.2.2)} \qquad (12)$$

With the values of [6] page 809, as lower bound of uniaxial strengths, (that could be regarded to be the strength at ~ 45° in fig. 8) is:

$$F_1 = \frac{1}{X} - \frac{1}{X'} = \frac{1}{700} - \frac{1}{400} \; ; \; F_2 = \frac{1}{Y} - \frac{1}{Y'} = \frac{1}{30} - \frac{1}{40} \; ; \; F_{11} = \frac{1}{XX'} = \frac{1}{700 \times 400} \; ;$$

$$F_{22} = \frac{1}{YY'} = \frac{1}{40 \times 30} \; ; \; \frac{1}{S^2} = \frac{1}{(103)^2} \text{ and } F_{12} = \pm \frac{1}{\sqrt{700 \,.\, 400 \,.\, 30 \,.\, 40}} = \pm 5{,}4 \,.\, 10^{-5}$$

as bounds. The value of S is taken from § 1.2.3., but the right value is not important in this case, because $1/S^2$ acts together with $2F_{12}$ and here: $2F_{12} + 1/S^2$ is a determining term.

The roots of eq. 12 (see also § 1.2.2) are given in fig. 11.

It is seen that for $F_{12} \approx 0$ there is a close fit to the Hankinson formula for tension that is supposed to be a good lower bound of measured values in [6] page 809.

For compression, $F_{12} = 0$ underestimates the Hankinson values, so for a precise fit the higher order terms $F_{221} \sigma_2^2 \sigma_1$ and $F_{112} \sigma_1^2 \sigma_2$ have to be used. However, the difference is to small to justify a more complex equation. It is not sure that the Hankinson formula for compression gives the points of first flow. Probably the higher order terms indicate that some plasticity was allowed in the tests.

The given bounds in fig. 11, connected with F_{12}, are also dependent on S. The relative lower value of S in [7] will shift those bounds and also the coëfficiënts in the Hankinson formula will be lower, (about 1,6 instead of 2).

The possibility of this lower coëfficiënt is also mentioned in [6].

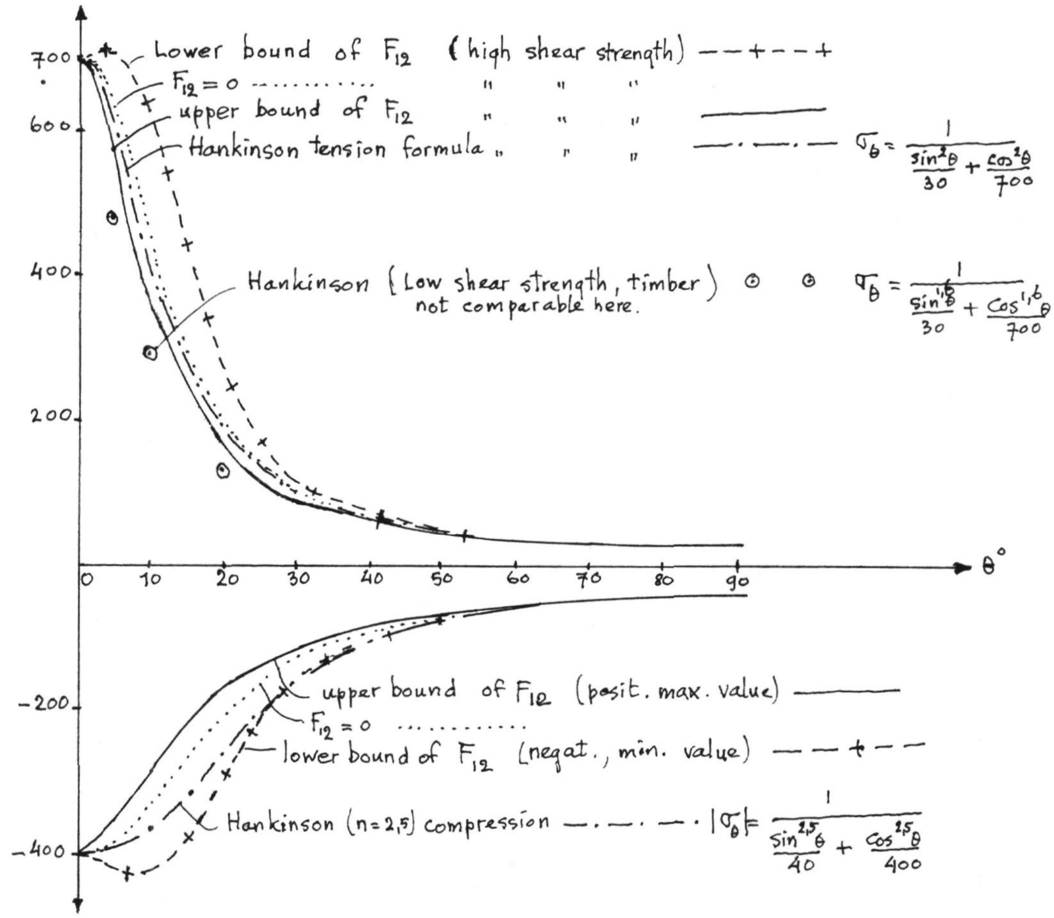


Fig. 11. Strength in kgf/cm²

1.3. Conclusion

It is demonstrated that a nearly exact representation of the failure surface of wood is given by the equation:

$$F_1 \sigma_1 + F_2 \sigma_2 + F_3 \sigma_3 + F_{11} \sigma_1^2 + 2F_{12} \sigma_1 \sigma_2 + 2F_{13} \sigma_1 \sigma_3 + F_{22} \sigma_2^2 +$$

$$+ 2F_{23} \sigma_2 \sigma_3 + F_{33} \sigma_3^2 + F_{44} \sigma_4^2 + F_{55} \sigma_5^2 + F_{66} \sigma_6^2 + 3F_{166} \sigma_1 \sigma_6^2 +$$

$$+ 3F_{112} \sigma_1^2 \sigma_2 + 3F_{221} \sigma_2^2 \sigma_1 = 1 \tag{13}$$

The value of F_{166} is a quick damping term with axis rotation and only important if fracture is surely in the radial plane. In practical applications, this can not be assured and this local strength increase has to be neglected. Also the influence of the third order terms F_{112}, F_{221} is to small to justify a more complicated equation (and bounds) and these terms are probably due to some allowed plasticity in the compression tests.

It is not known if this influence remains small in 3-axial test conditions. The roots λ of eq. (8) or (13) with the general value of a_3 (see §1.2.5)

$$a_3 = 3F_{166}\, S_1\, S_6^2 + 3F_{122}\, S_1\, S_2^2 + 3F_{211}\, S_2\, S_1^2$$

(if a small quantity), can exist of two small negative equal roots and a great positive one. So there is no theoretical exclusion of a high 3-axial strength.

Tests have to be done with unequal σ_1, σ_2 and σ_3 to measure these independent material properties.

Thus far, a strength increase is not apparant for tri-axial strength (see also § 2.4) and as a good approximation eq. (13) becomes:

$$F_1\,\sigma_1 + F_2\,\sigma_2 + F_3\,\sigma_3 + F_{11}\,\sigma_1^2 + 2F_{12}\,\sigma_1\,\sigma_2 + 2F_{13}\,\sigma_1\,\sigma_3 + F_{22}\,\sigma_2^2 + $$
$$+ 2F_{23}\,\sigma_2\,\sigma_3 + F_{33}\,\sigma_3^2 + F_{44}\,\sigma_4^2 + F_{55}\,\sigma_5^2 + F_{66}\,\sigma_6^2 = 1 \qquad (14)$$

For practical applications, the directions of the weakest plane $\perp$ grain, is not known in the structure and a lower bound criterion has to be used. It is shown in § 1.2.1. and 1.2.2. that with a lower bound on the tensile strength perpendicular to the grain, the quantities in the plane $\perp$ grain (here chosen as 2-3-plane) get the isotropic form, and F_{23} can be disregarded, so eq. (13) becomes:

$$\left(\frac{1}{X} - \frac{1}{X'}\right)\sigma_1 + \left(\frac{1}{Y} - \frac{1}{Y'}\right)(\sigma_2 + \sigma_3) + \frac{\sigma_1^2}{XX'} + 2F_{12}\,(\sigma_1\,\sigma_2 + \sigma_1\,\sigma_3) + \frac{\sigma_2^2 + \sigma_3^2 + 2\sigma_4^2}{YY'} +$$
$$+ \frac{1}{S^2}\,(\sigma_5^2 + \sigma_6^2) = 1 \qquad (14)$$

In § 1.2.5 it is shown that for small clear specimens and high shear strength (// grain), F_{12} can be neglected so eq. (14) can be:

$$\left(\frac{1}{X} - \frac{1}{X'}\right)\sigma_1 + \left(\frac{1}{Y} - \frac{1}{Y'}\right)(\sigma_2 + \sigma_3) + \frac{\sigma_1^2}{XX'} + \frac{\sigma_2^2 + \sigma_3^2 + 2\sigma_4^2}{YY'} + \frac{1}{S^2}\,(\sigma_5^2 + \sigma_6^2) = 1 \qquad (15)$$

In fig. 12 eq. (15) is given for only σ_1 and σ_2 ($\sigma_3 = \sigma_4 = \sigma_6 = 0$) in comparison with the usual applied Norris equations.

Eq. (15) lies closer to the older Norris equations, based on the Henkey-von Mises-theory and applied for plywood and for wood in the U.S.A. and

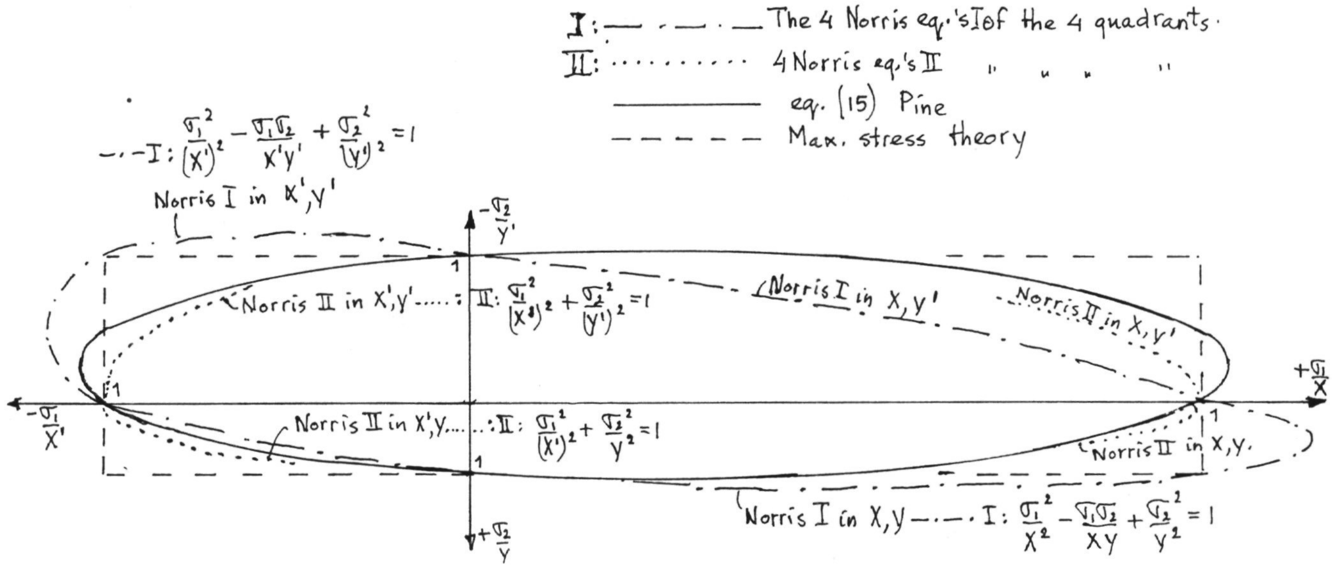


Fig. 12. Failure surface for $\sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0$.

the European (C.I.B.) code:

$$\frac{\sigma_1^2}{X^2} + \frac{\sigma_2^2}{Y^2} + \frac{\sigma_6^2}{S^2} = 1 \; ; \; \frac{\sigma_1^2}{(X')^2} + \frac{\sigma_2^2}{(Y')^2} + \frac{\sigma_6^2}{(S')^2} = 1 \quad \text{etc.,}$$

than to later proposed equations for wood, what are known to be not entirely adequate.

Eq. (15) is an extension of the strength criterion to 3 dimensions.

The general form of eq. (15), independent of the choise of the coordinate system is:

$$F_1' \sigma_1 + F_2' \sigma_2 + F_3' \sigma_3 + F_6' \sigma_6 + F_{11}' \sigma_1^2 + F_{22}' \sigma_2^2 + F_{33}' \sigma_3^2 + F_{44}' \sigma_4^2 +$$

$$+ F_{55}' \sigma_5^2 + F_{66}' \sigma_6^2 + 2F_{12}' \sigma_1 \sigma_2 + 2F_{23}' \sigma_2 \sigma_3 + 2F_{31}' \sigma_3 \sigma_1 + 2F_{16}' \sigma_1 \sigma_6 +$$

$$2F_{26}' \sigma_2 \sigma_6 + 2F_{36}' \sigma_3 \sigma_6 + 2F_{45}' \sigma_4 \sigma_5 = 1 \qquad (15')$$

2. Physical failure criteria

2.1. Discussion of criteria based on plasticity theory

2.1.1. Potential energy function

For yield phenomena, occurring e.g. in wood in compression, an extension of the isotropic theory is known from Hill.
He postulated the existence of a quadratic plastic stress potential (potential energy function) that had to be orthogonal and symmetric. This leads to equal strengths in positive- and negative-direction and no yield for hydrostatic stresses. This is in general not true for anisotropy because for hydrostatic $\sigma_I = \sigma_{II} = \sigma_{III}$; $\varepsilon_I \neq \varepsilon_{II} \neq \varepsilon_{III}$, and yield remains possible. Hoffmann [3] modified Hill's theory by adding linear terms to account for differences between tensile- and compressive strength.
The isotropic equivalence is the von Mises-Sleicher hypothesis that the second invariant of the deviatior stress tensor is not constant but a function of the mean stress (used for materials with Bauschinger effects). For isotropy it is the same to state that the critical distorsional energy is a function of the mean stress in stead of a constant value (as in the Henkey criterion). For anisotropy there is not such connection because the coupling of strengths need not be the same as given by the deviator stresses, so the Hill- and Hoffmann-equations are special cases of orthotropic strengths.
The Hill equation: $2f(\sigma) = A_1 (\sigma_2 - \sigma_3)^2 + A_2 (\sigma_1 - \sigma_2)^2 + A_3 (\sigma_1 - \sigma_3)^2 + 2A_4 \sigma_4^2 + 2A_5 \sigma_5^2 + 2A_6 \sigma_6^2 = 1$ has 6 constants and the surface is determined by the 3 principal yield stresses (as for isotropy) and also by the 3 directions of the principal strengths with respect to the material axes because these strengths are not necessarily along the material axes.
The equation contains a number of conditions.
Because of orthotropic symmetry of the material, the positive and negative shear strengths along the material axes are equal. This is given in the last 3 terms of the Hill equation. The first 3 terms contain 3 conditions of equal yield stress in tension and compression, and 3 oriëntations of the surface by the given values of the coupling terms of the normal stresses. In other words: a general quadratic orthotropic surface is determined by 12 constants.
These are the nine independent strength components (3 uniaxial tensile strengths; 3 uniaxial compressive strengths; 3 pure shear strengths)

and the 3 angles of oriëntation of the orthogonal surface with respect to the material axes.

The Hoffmann equation: $B_1 (\sigma_2 - \sigma_3)^2 + B_2 (\sigma_1 - \sigma_3)^2 + B_3 (\sigma_1 - \sigma_2)^2 + B_4 \sigma_3 + B_5 \sigma_2 + B_6 \sigma_1 + B_7 \sigma_6^2 + B_8 \sigma_5^2 + B_9 \sigma_4^2 = 1$ has 9 constants because now tension- and compression-strengths are different. The 3 special oriëntations of the Hill surface are however, retained.

As seen before, there is no coercive reason to do this. So all the 12 constants of the general form of a quadratic orthotropic surface

$$\sigma_1\left(\frac{1}{X} - \frac{1}{X'}\right) + \sigma_2\left(\frac{1}{Y} - \frac{1}{Y'}\right) + \sigma_3\left(\frac{1}{Z} - \frac{1}{Z'}\right) + \frac{\sigma_1^2}{XX'} + \frac{\sigma_2^2}{YY'} + \frac{\sigma_3^2}{ZZ'} + 2F_{12}\,\sigma_1\,\sigma_2 +$$

$$+ 2F_{23}\,\sigma_2\,\sigma_3 + 2F_{13}\,\sigma_1\,\sigma_3 + \frac{\sigma_4^2}{S_4^2} + \frac{\sigma_5^2}{S_5^2} + \frac{\sigma_6^2}{S_6^2} = 1$$

are independent material properties, and beside the strengths X, X', Y, ... etc, also the values of F_{12}, F_{23}, F_{13} have to be measured.

The potential energy function can be found by the principle of virtual works: $\phi = \delta W^{plast.} = \delta W^{extern.} + \delta W^{elast.}$, variating $\delta\varepsilon^P$ as virtual plastic deformation ($\varepsilon_i^P = \lambda_i\, \varepsilon^P$) and letting $\varepsilon^P \to 0$ for beginning of flow. Optimization of the function with respect to the displacements (λ_i) gives an unique, energetically feasible value of the starting of yielding.

Another approach is known from thermodynamics.

It is demonstrated there, that at flow, for sufficient small variations to get a linear irriversible proces, and Onsagers principle is appropriate, a function ψ exists so that:

$$\sigma_{ij} = \frac{\partial\psi}{\partial \dot{e}_{ij}^P} \quad \text{(see p.e. [10])}$$

The inverse relation is the plactic potential function: θ, being identical to the yield function at flow (for an isotherm proces) and for wich:

$$d\dot{e}_{ij}^P = \frac{\partial\theta}{\partial\sigma_{ij}} \cdot d\lambda$$

the normality rule applies.

2.1.2. Distortional energy theory

An extension of the distortional energy theory has been given by Norris for a special form of orthotropy.

With the scheme of the material as rectangular prismatic voids with isotropic walls, he calculated the distorsional energy for proportional loading and found 3 Mises type equations for each set of walls:

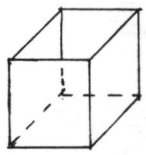


Fig. 13.

$$\frac{\sigma_1^2}{X^2} - \frac{\sigma_1 \sigma_2}{XY} + \frac{\sigma_2^2}{Y^2} + \frac{\sigma_6^2}{S_6^2} = 1 \; ; \; \frac{\sigma_2^2}{Y^2} - \frac{\sigma_2 \sigma_3}{YX} + \frac{\sigma_3^2}{X^2} + \frac{\sigma_4^2}{S_4^2} = 1 \; ;$$

$$\frac{\sigma_3^2}{Z^2} - \frac{\sigma_3 \sigma_1}{ZX} + \frac{\sigma_1^2}{Z^2} + \frac{\sigma_5^2}{S_5^2} = 1$$

or for plane stress:

$$\frac{\sigma_1^2}{X^2} + \frac{\sigma_2^2}{Y^2} - \frac{\sigma_1 \sigma_2}{XY} + \frac{\sigma_6^2}{S_6^2} = 1 \; ; \; \frac{\sigma_2^2}{Y^2} = 1 \; ; \; \frac{\sigma_1^2}{X^2} = 1$$

(If: $X = Y = Z = \sqrt{3} \times S_4 = \sqrt{3} \times S_5 = \sqrt{3} \times S_6 = \sigma_o$ = isotropic Henkey yield criterium).

Because there is a difference in compressional- and tensional-strengths we have to assume different critical energies for tension and compression and the only right interpretation of the Norris equations is to give different equations for each stress-quadrant. For instance for compression-tension:

$$\frac{\sigma_1^2}{X^2} - \frac{\sigma_1 \sigma_2}{XY'} + \frac{\sigma_2^2}{(Y')^2} + \frac{\sigma_6^2}{(S')^2} = 1 \text{ etc.}$$

This is given in fig. 12 for $\sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0$.

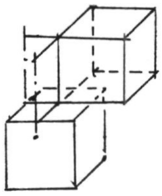
An experimental verification of this difference for tension and compression is p.e. given in [14].

As an extension of the Norris model we can assume an armature in the walls along the material axes, not interacting with each others like in concrete. Adding the energies of the armature we get the more general form:

$$\frac{\sigma_1^2}{X^2} + F_{12}\,\sigma_1\,\sigma_2 + \frac{\sigma_2^2}{Y^2} + \frac{\sigma_6^2}{S_6^2} = 1 \text{ etc.}$$

If we now assume initial stresses in the armature to give the material equal strength for tension and compression (e.g. initial stress $\sigma_i = \frac{X-Y}{2}$; tensile strength $\sigma_t = \frac{X+Y}{2} = -\sigma_{compr.} \rightarrow \sigma_t + \sigma_i = X$; $\sigma_i - \sigma_c = -Y$), we have to subtract these fictive stresses in the energy equations, and get the form:

$$\sigma_1 \left(\frac{1}{X} - \frac{1}{X'}\right) + \sigma_2 \left(\frac{1}{Y} - \frac{1}{Y'}\right) + \frac{\sigma_1^2}{XX'} + \frac{\sigma_2^2}{YY'} + F_{12}\,\sigma_1\,\sigma_2 + \frac{\sigma_6^2}{S_6^2} = 1$$



If there are other interactions of walls we finally get the three-dimensional equations like eq. 14 (§ 1) to be the critical distorsional energy equation for an orthotropic material like wood.

Fig. 14.

2.2. Hardening rules

Wood under compression exhibits plastic flow properties.
For tension, under certain operating conditions (e.g. impact), elastic deformation and brittle failures are more common.
However, for the usual loading conditions, the range of stable crack propagation is large enough to make an elastic-plastic description possible for tension.
For this reason, the limit analysis methods are in general applicable to wood (equilibrium method, etc.).
Because there is more plasticity in compression than in tension, the yield surface gets not only an expansion by hardening (like isotropic hardening) but also a translation (like kinematic hardening) and, as can be seen in fig. 9, also the shape of the surface changes. But the surface remains determined by the 12 independent strength components and it is necessary to know the hardening properties of these strength components.
Giving the loading function: $2f(\{\sigma\}, \sigma_u) = F_i\,\sigma_i + F_{ij}\,\sigma_i\,\sigma_j = 1$, with $\{\sigma\}$ the stress vector and σ_u the yield constants (X, Y, X', F_{12}, S_4 etc.) the relation between stresses and total strains:

$\{d\sigma\} = [S_{ep}]\{d\varepsilon\}$ can be found.

$\{d\varepsilon\} = \{d\varepsilon^e\} + \{d\varepsilon^p\}$ with an elastic- and a plastic part

Because of the associative flow rule, wich states that the plastic strain increment is perpendicular to the yield surface f: $d\varepsilon^p = d\lambda \left\{\frac{\partial f}{\partial \sigma}\right\}$

$\{d\sigma\} = [S_e]\,\{d\varepsilon^e\}$ is the elastic part so:

$$\{d\varepsilon\} = [S_e]^{-1}\,\{d\sigma\} + d\lambda\,\left\{\frac{\partial f}{\partial \sigma}\right\}$$

or, on multiplication by $\left\{\frac{\partial f}{\partial \sigma}\right\}^T\,[S_e]$:

$$\{\frac{\partial f}{\partial \sigma}\}^T [S_e]\{d\varepsilon\} = \{\frac{\partial f}{\partial \sigma}\}^T \{d\sigma\} + d\lambda \{\frac{\partial f}{\partial \sigma}\}^T [S_e]\{\frac{\partial f}{\partial \sigma}\}$$

Now for flow $2f = 1$, and for no unloading $df = 0$, so

$$df = 0 = \{\frac{\partial f}{\partial \sigma}\}^T \{d\sigma\} + \frac{\partial f}{\partial \sigma_u} \cdot d\sigma_u \text{ or } \{\frac{\partial f}{\partial \sigma}\}^T \{d\sigma\} = -\frac{\partial f}{\partial \sigma_u} d\sigma_u = A\, d\lambda,$$

and the above equation becomes:

$$\{\frac{\partial f}{\partial \sigma}\}^T [S_e]\{d\varepsilon\} = (A + \{\frac{\partial f}{\partial \sigma}\}^T [S_e]\{\frac{\partial f}{\partial \sigma}\})\, d\lambda \text{ and } \{d\varepsilon\} \text{ is found from:}$$

$$\{d\varepsilon\} = [S_e]^{-1}\{d\sigma\} + \{\frac{\partial f}{\partial \sigma}\} \frac{\{\frac{\partial f}{\partial \sigma}\}^T [S_e]\{d\varepsilon\}}{A + \{\frac{\partial f}{\partial \sigma}\}^T [S_e] \frac{\partial f}{\partial \sigma}}$$

Multiplication by S_e and rearranging gives:

$$\{d\sigma\} = \left[[S_e] - \frac{[S_e]\{\frac{\partial f}{\partial \sigma}\}\{\frac{\partial f}{\partial \sigma}\}^T [S_e]}{A + \{\frac{\partial f}{\partial \sigma}\}^T [S_e]\{\frac{\partial f}{\partial \sigma}\}} \right] \{d\varepsilon\} = [S_{ep}]\{d\varepsilon\}$$

$A = -\frac{1}{d\lambda}\frac{\partial f}{\partial \sigma_u} d\sigma_u$ is determined by the measured hardening diagrams:
$d\sigma_u = H_u\, d\varepsilon_u$ with arctg (H) as slope of the $\sigma_u - \varepsilon_u$ diagram. Because $d\varepsilon_u$ is the plastic flow of the special case of an uniaxial loading, the normality rule must also be valid and

$$d\varepsilon_u = \left| d\lambda \frac{\partial f}{\partial \sigma_u} \right| = -d\lambda \frac{\partial f}{\partial \sigma_u} \text{ and so:}$$

$$A = -\frac{1}{d\lambda}\frac{\partial f}{\partial \sigma_u} H_u\, d\varepsilon_u = (\frac{\partial f}{\partial \sigma_u})^2 H_u$$

For a weak hardening case we can make the following approximation:

$$[S_{ep}] = \left[[S_e] - \frac{[S_e]\{\frac{\partial f}{\partial \sigma}\}\{\frac{\partial f}{\partial \sigma}\}^T [S_e]}{A + \{\frac{\partial f}{\partial \sigma}\}^T [S_e]\{\frac{\partial f}{\partial \sigma}\}} \right] \approx \left[\frac{A\,[S_e]}{A + \{\frac{\partial f}{\partial \sigma}\}^T [S_e]\{\frac{\partial f}{\partial \sigma}\}} \right] \approx$$

$$\approx \frac{A\,[S_e]}{\{\frac{\partial f}{\partial \sigma}\}^T [S_e]\{\frac{\partial f}{\partial \sigma}\}} \approx \frac{[S_e]\,(\frac{\partial f}{\partial \sigma_u})^2 H_u}{\{\frac{\partial f}{\partial \sigma}\}^T [S_e]\{\frac{\partial f}{\partial \sigma}\}} \approx [S_e] \times \text{scalar}$$

and $\{d\varepsilon\} \approx \{d\varepsilon^p\}$. So $\{d\sigma\}$ is found by reduced elastic stiffness factors, depending on the state of stress.

The values of H can be found by uni-axial tests in the main directions. For the compression test $\perp$ e.g. is: (σ_2 to $\sigma_6 = 0$):

$$2f = F_i\,\sigma_i + F_{ij}\,\sigma_i\,\sigma_j = \left(\frac{1}{X}-\frac{1}{X'}\right)\sigma_1 + \frac{\sigma_1^2}{XX'} + \left(\frac{1}{Y}-\frac{1}{Y'}\right)\sigma_2 + \frac{\sigma_2^2}{YY'} + \ldots\ \text{etc.}$$

$$2\,\frac{\partial f}{\partial\sigma_1}\bigg|_{\sigma_1=-X'} = \left[\frac{1}{X}-\frac{1}{X'}+\frac{2\sigma_1}{XX'}\right]_{\sigma_1=-X'} + \frac{1}{Y}-\frac{1}{Y'}+\frac{1}{Z}-\frac{1}{Z'} \approx -\frac{1}{X}-\frac{1}{X'}+\frac{1}{Y}-\frac{1}{Y'}$$

if Z is the strong direction.

$$\left\{\frac{\partial f}{\partial\sigma}\right\} \approx \begin{Bmatrix} -\frac{1}{2}\left(\frac{1}{X}+\frac{1}{X'}\right) \\ \frac{1}{2}\left(\frac{1}{Y}-\frac{1}{Y'}\right) \\ \sim 0 \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \text{ and: } [S_e]\left\{\frac{\partial f}{\partial\sigma}\right\} = \begin{Bmatrix} -\frac{S_{11}}{2}\left(\frac{1}{X}+\frac{1}{X'}\right)+\frac{S_{12}}{2}\left(\frac{1}{Y}-\frac{1}{Y'}\right) \\ -\frac{S_{12}}{2}\left(\frac{1}{X}+\frac{1}{X'}\right)+\frac{S_{22}}{2}\left(\frac{1}{Y}-\frac{1}{Y'}\right) \\ -\frac{S_{13}}{2}\left(\frac{1}{X}+\frac{1}{X'}\right)+\frac{S_{23}}{2}\left(\frac{1}{Y}-\frac{1}{Y'}\right) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \text{ with}$$

$$[S_e] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \rightarrow \left\{\frac{\partial f}{\partial\sigma}\right\}^T [S_e]\left\{\frac{\partial f}{\partial\sigma}\right\} = \frac{S_{11}}{4}\left(\frac{1}{X}-\frac{1}{X'}\right)^2 + $$

$$-\frac{S_{12}}{2}\left(\frac{1}{X}+\frac{1}{X'}\right)\times\left(\frac{1}{Y}-\frac{1}{Y'}\right)+\frac{1}{4}S_{22}\left(\frac{1}{Y}-\frac{1}{Y'}\right)^2$$

$$\left(\frac{\partial f}{\partial\sigma_u}\right)^2 H_u = \left[\frac{1}{4}\left(-\frac{\sigma_1}{X^2}-\frac{\sigma_1^2}{X^2X^1}\right)^2 \cdot H_X\right]_{\sigma_1=-X'} + \left[\frac{1}{4}\left(+\frac{\sigma_1}{X'^2}-\frac{\sigma_1^2}{X(X')^2}\right)^2 H_{X'}\right]_{\sigma_1=-X'} =$$

$$= \frac{1}{4}\left(\frac{X'}{X^2}-\frac{X'}{X^2}\right)^2 H_X + \frac{1}{4}\left(-\frac{1}{X'}-\frac{1}{X}\right)^2 H_{X'} = \frac{1}{4}\left(\frac{1}{X'}+\frac{1}{X}\right)^2 H_{X'} \rightarrow$$

$$\frac{\left(\frac{\partial f}{\partial\sigma_u}\right)^2 H_u}{\left\{\frac{\partial f}{\partial\sigma}\right\}^T [S_e]\left\{\frac{\partial f}{\partial\sigma}\right\}} = \frac{\frac{1}{4}\left(\frac{1}{X'}+\frac{1}{X}\right)^2 H_{X'}}{\frac{S_{11}}{4}\left(\frac{1}{X}+\frac{1}{X'}\right)^2 - \frac{S_{12}}{2}\left(\frac{1}{X}+\frac{1}{X'}\right)\times\left(\frac{1}{Y}-\frac{1}{Y'}\right) + \frac{1}{4}S_{22}\left(\frac{1}{Y}-\frac{1}{Y'}\right)^2} =$$

$$= \frac{H_{X'}}{S_{11}\ (1 - 2\ \frac{S_{12}}{S_{11}} \cdot (\frac{1}{Y} - \frac{1}{Y'})/(\frac{1}{X} + \frac{1}{X'}) + \frac{S_{22}}{S_{11}} \cdot (\frac{1}{Y} - \frac{1}{Y'})^2 / (\frac{1}{X} + \frac{1}{X'})^2\ )} = \frac{H_{X'}}{S_{11}\ (1-\gamma)}$$

$$\begin{Bmatrix} d\sigma_1 \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} = [S_e] . \frac{H_{X'}}{S_{11}\ (1-\gamma)} . \begin{Bmatrix} d\varepsilon_1 \\ d\varepsilon_2 \\ d\varepsilon_3 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} d\varepsilon_1 \\ d\varepsilon_2 \\ d\varepsilon_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = [S_e]^{-1} \begin{Bmatrix} d\sigma_1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \frac{S_{11}}{H_{X'}} (1-\gamma)$$

or:

$$d\varepsilon_1 = C_{11}\ d\sigma_1 \cdot \frac{S_{11}}{H_{X'}} (1-\gamma) \quad \text{with} \quad C_{11} = \begin{vmatrix} S_{22} & S_{23} \\ S_{23} & S_{33} \end{vmatrix} \Big/ \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{vmatrix}$$

From the measured values $\{d\varepsilon\}$; $[S_e]$; $d\sigma_1$, the value of $H_{X'}$ can be found using least squares with the other equations in $d\varepsilon_2$ and $d\varepsilon_3$ too.
It is seen that the determination of $[S_{ep}]$ is a very lengthy and laborious task, only suitable for digital computation.
Another possibility is the elastic-fully plastic approximation with:

$$\{d\varepsilon^p\} = d\lambda\ \{\frac{\partial F}{\partial \sigma}\} \quad (\text{so } A = H = 0)$$

In this case, for $\sigma_1 = -X'$ and other $\sigma_{i \neq 1} = 0$ is:

$$d\varepsilon_1 = d\lambda\ \left[(\frac{1}{X} - \frac{1}{X'} + \frac{2\sigma_1}{XX'})\right]_{\sigma_1 = -X'} = d\lambda\ (\frac{1}{X} - \frac{1}{X'} - \frac{2}{X}) = -d\lambda\ (\frac{1}{X'} + \frac{1}{X})$$

and:

$$d\varepsilon_2 = d\lambda\ \left[\frac{1}{Y} - \frac{1}{Y'} + \frac{2\sigma_2}{YY'}\right]_{\sigma_2 = 0} = d\lambda\ (\frac{1}{Y} - \frac{1}{Y'})$$

So $\frac{d\varepsilon_1}{d\varepsilon_2} = \frac{-(\frac{1}{X'} + \frac{1}{X})}{\frac{1}{Y} - \frac{1}{Y'}}$ is normal to the surface of fig. 12 in point:

$(\sigma_1 = -X'; \sigma_2 = 0)$ as expected.
It is intended to measure whether this approach is sufficient.

2.3. Some remarks on criteria based on fracture mechanics

The most simple criteria used for orhotropy are the maximum stress theory and the maximum strain theory.
The maximum strain theory, as extension of St. Venants theory leads to contradictions (see [2]).
The maximum stress theory states, for orthotropic materials, that the strength is reached, when any stress along the natural axes reaches its maximum value.
This theory neglects interaction of stresses and the domain where this can approximately be right is given by fracture mechanics because only the stresses in one plane are magnified by a flat crack and one single principal tensile strength may determine the total strength.
The strength is now a plane problem determined by Mohr's envelop. If we look at the maximum stresses lying along the crack boundary, the strength can be determined by the principal tensile stress, being the only magnified stress. So whatever the fracture criterion is, there is only one stress (others being neglectible) determining the strength. If we do the variable transformation of appendix 1, we can use the isotropic solution of the crack problem for orthotropic material.
For a crack propagating $\perp$ to the direction of the maximum tensile stress along the boundary litt. [12] p. 265 gives:

$$\frac{\tau^2}{(2\sigma_t)^2} + \frac{\sigma_y}{\sigma_t} = 1 \quad \text{or} \quad \left(\frac{\tau}{\tau_u}\right)^2 + \frac{\sigma_y}{\sigma_t} = 1$$

The form remains the same if we transform this back to the original variables. So for not collinear crack propagation depending on the uniaxial tensile strength (along the boundary of the flat elliptical crack) we have

$$\frac{K_I}{K_{Ic}} + \left(\frac{K_{II}}{K_{IIc}}\right)^2 = 1$$

This is measured by Chow and Woo [13] for a light wood species and is also measured in the radial plane of pine (see § 1, fig. 10: $\frac{\tau}{\tau_u} \approx \sqrt{1 - \frac{\sigma_1}{X}}$).
So it is not necessary to assume friction for the shear strength increase // by compression $\perp$ ([13] Jaeger, Keenan). It is noticed that, despite of some compression, failure can be in the opening mode, so the failure is not by shear (and friction) but by non-collinear crack propagation and "K_{IIc}" is dependent on K_{Ic} and is smaller than the real value of K_{IIc}

(This is often found in tests see [13] J.F. Murphy p.e.).
For collinear crack-propagation is: $\frac{\sigma_y}{\sigma_t} = 1$ and only for tension $\perp$ to the crack this will be a principal stress. So for combinations of shear and tension, both stresses are magnified and we have to know the failure criterion. For this, sometimes the maximum stress criterion:

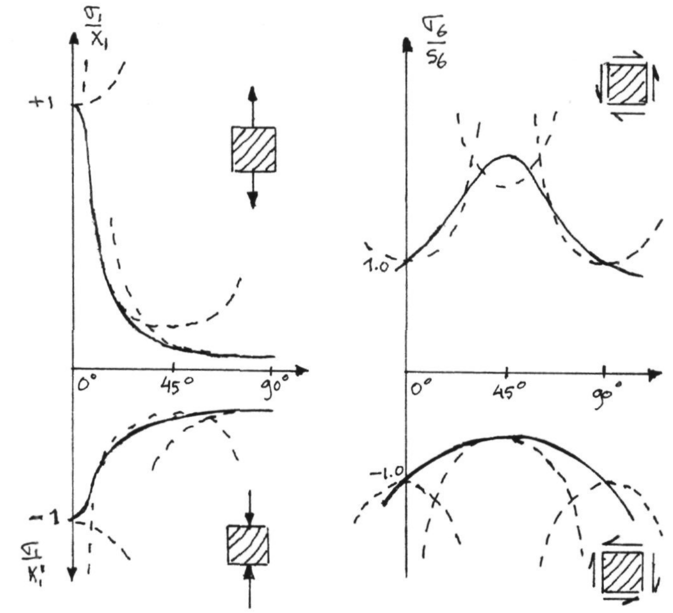
$$\frac{K_I}{K_{Ic}} \leq 1; \quad \frac{K_{II}}{K_{IIc}} \leq 1$$

is used, or the linear combination:

$$\frac{K_I}{K_{Ic}} + \frac{K_{II}}{K_{IIc}} \leq 1$$

to account for the smaller "K_{IIc}" of non-collinear crack propagation and to maintain the separated measured real K_{IIc} in this fracture criterion. However, the real K_{IIc} from collinear propagation due to pure shear along the grain can only be a local strength increase in timber because of faults, deviations of the grain directions, knots etc.
If it is assumed (as usual for wood) that the initial crack is in a plane along the grain (or $\perp$) and also the propagation is collinear (along the grain (or $\perp$) the fracture criterion for the region around the crack tip must be the same as the macro-criterion expressed in stresses in the material axes and must have the form of the (extended) plane Norris equations. (like eq. 14 §1)

However, non-collinear crack propagation is apparent and also the compression stress around the crack can be high, giving stress redistribution around the crack by layer-buckling etc., so using the concept of the critical strain-energy-density at the borders of the plastified areas, and assuming randomly oriënted cracks, we get a similar 3-dimensional, quadratic polynome as given in §1 and §2 as expansion of the failure criterion.
The maximum stress criterion can not be made entirely adequate as can be seen in the next schematic graphs of the strengths, and in fig. 12 (and 12').



Solid lines : polynomial equation

Dashed lines: maximum stress theory

Fig. 15.

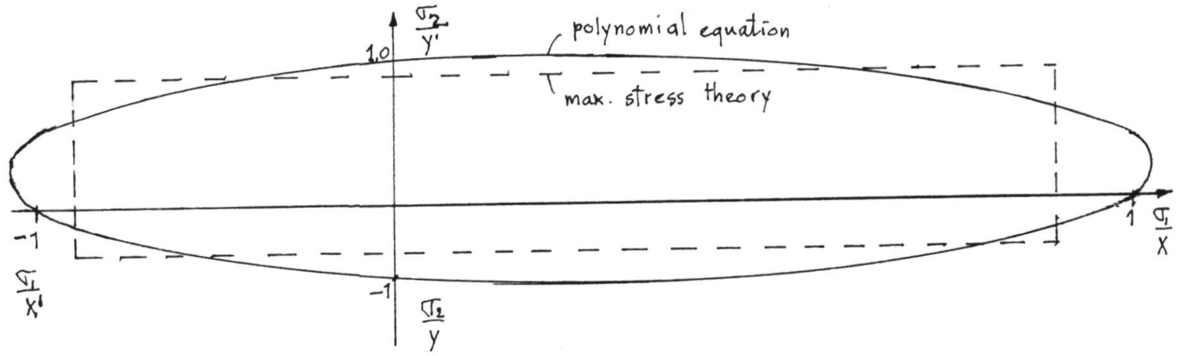


Fig. 12'.

2.4. Layer buckling

Interation equations of buckling contain linear and quadratic terms, p.e.:

$$\left(\frac{\tau}{\tau_{cr}}\right)^2 + \frac{\sigma_1}{X'_{cr}} + \frac{\sigma_2}{Y'_{cr}} = 1$$

similar as for crack propagation (§ 2.3).

However there will always be interaction with crack propagation and the buckled areas increase the "plastified" zone around the crack tips so a more general interaction equation (like eq. 14) is probable.
If buckling is the ultimate condition for compressional failure of wood, there will be no strong strength increase if $\varepsilon_I = \varepsilon_{II} = \varepsilon_{III}$, as expected from plasticity theory, because the real stresses in the cell walls are plane and one real ε is zero.

2.5. Viscous properties

Wood has viscous properties and is influenced by time, temperature, moisture etc.
All those influences can be taken into account in the strength parameters. So e.g. the uni-axial strength criterion becomes:

$$\sigma_1 \left(\frac{1}{X(t)} - \frac{1}{X'(t)}\right) + \frac{\sigma_1^2}{X(t) \, . \, X'(t)} = 1$$

where X and X' depend on time t, with possible different rates of creep and different long term strengths.
So far, there are no observations, that contradict this model.

2.6. Conclusion on physical failure criteria

It is demonstrated that the general orthotropic quadratic polynomial stress equation represents a potential energy function or the critical distortional energy function for an orthotropic material.
The polynomial expansion of that function θ will have the orthotropic basis for wood:

$$\theta \ (\sigma_1; \sigma_2; \sigma_3; \sigma_4^2; \sigma_5^2; \sigma_6^2) \text{ (see [9]; orthogonal planes } x_1 = 0; x_2 = 0)$$

and for a transverse isotropic approximation (or lower bound § 1.3):

$$\theta \ (\sigma_1; \sigma_2 + \sigma_3; \sigma_2 \sigma_3 - \sigma_4^2; \ \sigma_5^2 + \sigma_6^2; \text{det. } (\sigma_{ij})) \text{ in general or:}$$

$$\theta \ (I_1; I_2; I_3; \sigma_3; \sigma_4^2 + \sigma_5^2) \text{ with } I_1, I_2, I_3 \text{ the 3 stress invariants}$$

(symmetry about X_1-axis)
For plane stress: $\sigma_3 = \sigma_4 = \sigma_5 = 0$, the polynomial basis for both cases (orthotropic and transverse isotropic) is:

θ (σ_1; σ_2; σ_6^2) or in general:

$$\theta = C_1 \sigma_1 + C_2 \sigma_2 + \frac{1}{2} C_{11} \sigma_1^2 + \frac{1}{2} C_{22} \sigma_2^2 + C_{12} \sigma_1 \sigma_2 + \frac{1}{2} C_{66} \sigma_6^2$$

It is shown in § 1 (1.2.4. and 1.2.5.) that there is some influence of higher order terms, probably due to some non-linear elasticity and plasticity, and some deviation from orthogonal strength behaviour. The quadratic polynomium is an inscribed surface or lower bound of the strength. We can only expect that some function of the distorsional energy gets a critical value. It is known that for wood in the elastic stage there is also some deviation from orthotropic behaviour. For practice however, the assumption of orthotropic elastic- and plastic behaviour with critical distortional energy for flow is a sufficient approximation.
The best, we can regard θ as a potential energy function.
Because the polynomium is an expansion of the real yield surface, higher order terms are possible depending on the form of the surface (and flow rules). As shown above this surface will be a complicated function of all stress invariants.
From the model of non-collinear crack propagation of randomly oriënted cracks we can also expect to have one mean pure shearing strength in the main planes determined by the tensile strength near the crack tips (so $C_6 = C_{16} = C_{26} = 0$ but $C_{166} \neq 0$ etc., see 1.1.; 1.2.4. and 2.3.) but also to have an interaction between shear- and normal-strengths.
In general the conclusions of § 1.3. are confirmed.

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Appendix 1

Variable transformation for orthotripic plane problems

For plane stress is Hooke's law:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{11}} & -\frac{\nu_{12}}{E_{11}} & 0 \\ -\frac{\nu_{21}}{E_{22}} & \frac{1}{E_{22}} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

Introducing new coördinates as variable transformation:

$$x = x_1/\sqrt{\delta} \quad ; \; y = x_2 \sqrt{\delta}$$

the stresses and strains become

$$\sigma_{xx} = \sigma_{11}/\delta \; ; \; \sigma_{yy} = \sigma_{22}\,\delta \; ; \; \sigma_{xy} = \sigma_{12} \; ; \; \varepsilon_{xx} = \varepsilon_{11}\,\delta \; ; \; \varepsilon_{yy} = \varepsilon_{22}/\delta \; ;$$
$$\varepsilon_{xy} = \varepsilon_{12}$$

Hooke's law becomes:

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(K+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \qquad (1)$$

with: $\delta^4 = E_{11}/E_{12} = \nu_{12}/\nu_{21}$ (symmetry)

$$E = \sqrt{E_{11}\,E_{22}}$$

$$\nu = \sqrt{\nu_{12}\,\nu_{21}}$$

$$K = \frac{1}{2}\sqrt{E_{11}\,E_{22}}\left(\frac{1}{G_{12}} - \frac{\nu_{12}}{E_{11}} - \frac{\nu_{21}}{E_{22}}\right) \quad \text{with } \frac{\nu_{12}}{E_{11}} = \frac{\nu_{21}}{E_{22}}$$

(As indication for wood: $\delta \approx K \approx 2$; for isotropy: $\delta = K = 1$).

For plane stress and:

$$E = \sqrt{\frac{E_{11}\ E_{22}}{(1-\nu_{13}\ \nu_{31})(1-\nu_{23}\ \nu_{32})}};\ \nu = \sqrt{\frac{(\nu_{12}+\nu_{13}\ \nu_{32})(\nu_{21}+\nu_{23}\ \nu_{31})}{(1-\nu_{13}\ \nu_{31})(1-\nu_{23}\ \nu_{32})}};$$

$$\delta^4 = \frac{E_{11}}{E_{22}} \times \frac{1-\nu_{23}\ \nu_{32}}{1-\nu_{13}\ \nu_{33}};\ K = \frac{1}{2}\ E\ (\frac{1}{G_{12}} - \frac{\nu_{12}+\nu_{13}\ \nu_{32}}{E_{11}} - \frac{\nu_{21}+\nu_{23}\ \nu_{31}}{E_{22}})$$

for plane strain.

The Airy stress function ϕ is the same for the original- and transformed variables, so for equilibrium:

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}\ ;\ \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}\ ;\ \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\ \partial y} \qquad (2)$$

and for compatibility:

$$(\frac{\partial^4\phi}{\partial x^4} + 2K\frac{\partial^4\phi}{\partial x^2\ \partial y^2} + \frac{\partial^4\phi}{\partial y^4}) = 0 \qquad (3)$$

Now eq. (1) can be written in the isotropic form:

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon'_{xy}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma'_{xy}\end{bmatrix}$$

With $\sigma'_{xy} = \sigma_{xy}$ and $\varepsilon'_{xy} = \varepsilon_{xy}\frac{1+\nu}{K+\nu}$ the isotropic solution of ϕ is a lower bound because the solution satisfies the equilibrium conditions, but is not compatible. With $\varepsilon'_{xy} = \varepsilon_{xy}$ and $\sigma'_{xy} = \frac{K+\nu}{1+\nu}\sigma_{xy}$, the solution satisfies compatibility but not the equilibrium conditions, thus is an upper bound solution.

For both cases the normal stresses are the same. So if there is a maximum normal ultimate stress criterium, the isotropic value of ϕ gives a possible solution what is the right value of the ultimate state (with equal upper- and lower bounds).

The calculated ultimate state differs an internal equilibrium system from the real ultimate state without affecting the ultimate value.

The same can be stated for an ultimate shear stress criterion.

The internal equilibrium system affects only the normal stresses in this case.
So for a proper geometric- and material-transformation the solution of the crack-problem of an orthotropic material is the same as for an isotropic material.